

ZEROS OF FUNCTIONS IN BERGMAN-TYPE HILBERT SPACES OF DIRICHLET SERIES

OLE FREDRIK BREVIG

ABSTRACT. For a real number α the Hilbert spaces \mathcal{D}_α consists of those Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ for which $\sum_{n=1}^{\infty} |a_n|^2/[d(n)]^\alpha < \infty$, where $d(n)$ denotes the number of divisors of n . We extend a theorem of Seip on the bounded zero sequences of functions in \mathcal{D}_α to the case $\alpha > 0$. Generalizations to other weighted spaces of Dirichlet series are also discussed, as are partial results on the zeros of functions in the Hardy spaces of Dirichlet series \mathcal{H}^p , for $1 \leq p < 2$.

1. INTRODUCTION

Let $d(n)$ denote the divisor function let α be a real number. We are interested in the following Hilbert spaces of Dirichlet series:

$$\mathcal{D}_\alpha = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : \|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{[d(n)]^\alpha} < \infty \right\}.$$

The functions of \mathcal{D}_α are holomorphic in $\mathbb{C}_{1/2}^+ = \{s = \sigma + it : \sigma > 1/2\}$. Bounded Dirichlet series are almost periodic, and this implies that they have either no zeros or infinitely many zeros, as observed by Olsen and Seip in [10]. This leads us to restrict our investigations to bounded zero sequences for spaces of Dirichlet series. In [13], Seip studied bounded zero sequences for \mathcal{D}_α , when $\alpha \leq 0$. This includes the Hardy-type ($\alpha = 0$) and Dirichlet-type ($\alpha < 0$) spaces. The topic of the present work is the Bergman-type spaces ($\alpha > 0$).

Let us therefore introduce the weighted Bergman spaces in the half-plane, A_β . For $\beta > 0$, these spaces consists of functions F which are holomorphic in $\mathbb{C}_{1/2}^+$ and satisfy

$$\|F\|_{A_\beta} = \left(\int_{\mathbb{C}_{1/2}^+} |F(s)|^2 \left(\sigma - \frac{1}{2} \right)^{\beta-1} dm(s) \right)^{\frac{1}{2}} < \infty.$$

It was shown by Olsen in [9] that the local behavior of the spaces \mathcal{D}_α are similar to the spaces A_β , where $\beta = 2^\alpha - 1$. This relationship between α and β will be retained throughout this paper.

For any space of holomorphic functions \mathcal{C} on some domain, we let $Z(\mathcal{C})$ denote the set of sequences in the domain such that there is some $F \in \mathcal{C}$ vanishing on S , taking into account multiplicities.

A result proved by Horowitz in [6] shows that if $\mathcal{C} = A_\beta$ we may assume that F vanishes precisely on $S \in Z(A_\beta)$, i.e. F has no extraneous zeros in $\mathbb{C}_{1/2}^+$. We will exploit this fact to prove our main result.

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Theorem 1. *Suppose $S = (\sigma_j + it_j)$ is a bounded sequence of points in $\mathbb{C}_{1/2}^+$ and that $\alpha > 0$. Then there is a non-trivial function in \mathcal{D}_α vanishing on S if and only if $S \in Z(A_\beta)$.*

The “only if” part follows from the local embedding of \mathcal{D}_α into A_β of Theorem 1 and Example 4 from [9]. To prove the “if” part, we will adapt the methods of [13], where an analogous result for $\alpha \leq 0$ was obtained.

The “if” part can essentially be split into two steps. The first step is a discretization lemma, which depends on the properties of \mathcal{D}_α — or rather the weights $[d(n)]^\alpha$. The second step is an iterative scheme, where the properties of A_β become more prominent.

Comparing this with [13], the first step is somewhat harder, since we require very precise estimates on the weights as α grows to infinity. The second step is considerably easier, mainly due to the fact that the norms of A_β are easier to work with than those of the Dirichlet spaces used in [13].

2. PROOF OF THEOREM 1

We begin with the Paley–Wiener representation of functions $F \in A_\beta$, and seek to construct a Dirichlet series $f \in \mathcal{D}_\alpha$ which approximates F .

Lemma 2 (Paley–Wiener Representation). *A_β is isometrically isomorphic to*

$$L_\beta^2 = \left\{ \phi \text{ measurable on } [0, \infty) : \|\phi\|_{L_\beta^2}^2 = \frac{2\pi\Gamma(\beta)}{2^\beta} \int_0^\infty |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta} < \infty \right\},$$

under the Laplace transformation

$$F(s) = \int_0^\infty \phi(\xi) e^{-(s-1/2)\xi} d\xi.$$

Proof. A proof can be found in [2]. □

The other ingredient needed for the discretization lemma is estimates on the growth of $[d(n)]^\alpha$. We will partition the integers into blocks and use an average order type estimate.

Lemma 3. *Let α be a real number and $0 < \gamma < 1$. Then*

$$(1) \quad \sum_{j^\gamma \leq \log n \leq (j+1)^\gamma} \frac{[d(n)]^\alpha}{n} \asymp j^{\gamma 2^\alpha - 1},$$

as $j \rightarrow \infty$. The implied constants may depend on α and γ .

Proof. We will first assume that 2^α is not an integer. To obtain the required estimate, we apply the precise form of a formula by Ramanujan [11] and Wilson [14]: For any real number α and any integer $\nu > 2^\alpha - 2$, we have

$$(2) \quad D_\alpha(x) = \sum_{n \leq x} [d(n)]^\alpha = x(\log x)^{2^\alpha - 1} \left(\sum_{\lambda=0}^{\nu} \frac{A_\lambda}{(\log x)^\lambda} + \mathcal{O}\left(\frac{1}{(\log x)^{\nu+1}}\right) \right).$$

Fix ν such that $\nu > 2^\alpha - 1$ and $\nu > 1/\gamma - 1$. We use Abel summation to rewrite

$$(3) \quad \sum_{y < n \leq x} \frac{[d(n)]^\alpha}{n} = \frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} + \int_y^x \frac{D_\alpha(z)}{z^2} dz.$$

By using (2) and the fact that $2^\alpha - 1 - \nu < 0$ we perform some standard calculations to estimate

$$\begin{aligned} \frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} &= \sum_{\lambda=0}^{\nu} A_\lambda \left((\log x)^{2^\alpha-1-\lambda} - (\log y)^{2^\alpha-1-\lambda} \right) + \mathcal{O} \left((\log y)^{2^\alpha-2-\nu} \right), \\ \int_y^x \frac{D_\alpha(z)}{z^2} dz &= \sum_{\lambda=0}^{\nu} \frac{A_\lambda}{2^\alpha-\lambda} \left((\log x)^{2^\alpha-\lambda} - (\log y)^{2^\alpha-\lambda} \right) + \mathcal{O} \left((\log y)^{2^\alpha-1-\nu} \right). \end{aligned}$$

Let us now take $x = \exp((j+1)^\gamma)$ and $y = \exp(j^\gamma)$. For any exponent η it is clear that

$$(\log x)^\eta - (\log y)^\eta = \gamma \eta j^{\gamma\eta-1} \left(1 + \mathcal{O} \left(\frac{1}{j} \right) \right).$$

Hence we have

$$\begin{aligned} \frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} &\asymp \sum_{\lambda=0}^{\nu} A_\lambda (\gamma(2^\alpha-1-\lambda)) j^{\gamma(2^\alpha-1-\lambda)-1} + \mathcal{O} \left(j^{\gamma(2^\alpha-2-\nu)} \right), \\ \int_y^x \frac{D_\alpha(z)}{z^2} dz &\asymp \sum_{\lambda=0}^{\nu} A_\lambda j^{\gamma(2^\alpha-\lambda)-1} + \mathcal{O} \left(j^{\gamma(2^\alpha-1-\nu)} \right). \end{aligned}$$

We combine these estimates with (3) to obtain

$$(4) \quad \sum_{j^\gamma \leq \log n \leq (j+1)^\gamma} \frac{[d(n)]^\alpha}{n} \asymp j^{\gamma 2^\alpha-1} \left(1 + \sum_{\lambda=1}^{\nu} \frac{B_\lambda}{j^{\gamma\lambda}} + \mathcal{O} \left(\frac{1}{j^{\gamma 2^\alpha-1-\gamma(2^\alpha-1-\nu)}} \right) \right),$$

which proves (1) since $\nu > 1/\gamma - 1$. By continuity on both sides of (4), the assumption that 2^α is not an integer may be dropped. \square

The parameter $0 < \gamma < 1$ will be used to control the “block size” of our partition of the integers. It will become apparent that as α grows to infinity, we must be able to let γ tend to 0. In [13] it was sufficient to have a similar estimate only for $1/2 < \gamma < 1$.

Lemma 4 (Discretization Lemma). *Let $\alpha > 0$ and let N be a sufficiently large positive integer. Then there exists positive constants A and B (depending on α , but not N) such that the following holds: For every function $\phi \in L_\beta^2$ supported on $[\log N, \infty)$, there is a function of the form*

$$f(s) = \sum_{n=N}^{\infty} \frac{a_n}{n^s}$$

in \mathcal{D}_α such that $\|f\|_{\mathcal{D}_\alpha} \leq A\|\phi\|_{L_\beta^2}$. Moreover, f may be chosen so that

$$\Phi(s) = \int_{\log N}^{\infty} \phi(\xi) e^{-(s-1/2)\xi} d\xi - f(s)$$

enjoys the estimate

$$|\Phi(s)| \leq B|s-1/2|N^{-\sigma+1/2}(\log N)^{-1}\|\phi\|_{L_\beta^2},$$

in $\mathbb{C}_{1/2}^+$.

Proof. Let $\gamma = 2/(4+2^\alpha)$ and let J be the largest integer smaller than $(\log(N))^{1/\gamma}$. For $j \geq J$, let n_j be the smallest integer n such that $e^{j^\gamma} \leq n$. When γ is small it is possible that $n_j = n_{j+1}$. This can be avoided by taking N sufficiently large. Set $\xi_{n_j} = j^\gamma$ and for $n_j < n \leq n_{j+1}$ iteratively choose ξ_n such that

$$(5) \quad \frac{\xi_{n+1}^{\beta+1} - \xi_n^{\beta+1}}{\beta+1} = A_j \frac{[d(n)]^\alpha}{n},$$

where A_j is chosen so that $\xi_{n_{j+1}} = (j+1)^\gamma$. Clearly, Lemma 1 implies that A_j is bounded as $j \rightarrow \infty$. Let us set

$$a_n = \sqrt{n} \int_{\xi_n}^{\xi_{n+1}} \phi(\xi) d\xi.$$

A simple computation using the Cauchy–Schwarz inequality shows that

$$|a_n|^2 = n \left| \int_{\xi_n}^{\xi_{n+1}} \phi(\xi) d\xi \right|^2 \leq n \cdot \frac{\xi_{n+1}^{\beta+1} - \xi_n^{\beta+1}}{\beta+1} \int_{\xi_n}^{\xi_{n+1}} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta}.$$

In view of (5) it is clear that $\|f\|_{\mathcal{D}_\alpha} \leq A\|\phi\|_{L_\beta^2}$. Now, if $n_j \leq n \leq n_{j+1}$ and $\xi \in [\xi_{n_j}, \xi_{n_{j+1}}]$ we see that

$$(6) \quad \left| e^{-(s-1/2)} - n^{-(s-1/2)} \right| \leq N^{-\sigma+1/2} |s-1/2| j^{\gamma-1}.$$

Then, by (6) and the Cauchy–Schwarz inequality

$$|\Phi(s)| \leq N^{-\sigma+1/2} |s-1/2| \sum_{j=J}^{\infty} j^{\gamma-1} \sum_{n=n_j}^{n_{j+1}-1} \left(\frac{\xi_{n+1}^\beta - \xi_n^\beta}{\beta} \right)^{\frac{1}{2}} \left(\int_{\xi_n}^{\xi_{n+1}} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta} \right)^{\frac{1}{2}}.$$

By using the Cauchy–Schwarz inequality again with (5) we get

$$|\Phi(s)| \ll N^{-\sigma+1/2} |s-1/2| \sum_{j=J}^{\infty} j^{\gamma-1} \left(\sum_{n=n_j}^{n_{j+1}-1} \frac{[d(n)]^\alpha}{n} \right)^{\frac{1}{2}} \left(\int_{\xi_{n_j}}^{\xi_{n_{j+1}}} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta} \right)^{\frac{1}{2}}.$$

Now Lemma 3 and the Cauchy–Schwarz inequality yield

$$|\Phi(s)| \ll N^{-\sigma+1/2} |s-1/2| \left(\sum_{j=J}^{\infty} j^{(2+2^\alpha)\gamma-3} \right)^{\frac{1}{2}} \left(\int_{\log N}^{\infty} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta} \right)^{\frac{1}{2}}.$$

The series converges since $\gamma < 2/(2+2^\alpha)$. The proof is completed by a standard estimate of the convergent series,

$$\left(\sum_{j=J}^{\infty} j^{(2+2^\alpha)\gamma-3} \right)^{\frac{1}{2}} \ll (\log N)^{((2+2^\alpha)\gamma-2)/(2\gamma)} = (\log N)^{-1},$$

where we used that $J \asymp (\log N)^{1/\gamma}$. \square

The final result needed for the iterative scheme is the following simple lemma on the $\bar{\partial}$ -equation. We omit the proof, which is obvious.

Lemma 5. *Suppose g is a continuous function on $\mathbb{C}_{1/2}^+$, supported on*

$$\Omega(R, \tau) = \{s = \sigma + it : 1/2 \leq \sigma \leq 1/2 + \tau, -R \leq t \leq R\}.$$

Then

$$u(s) = \frac{1}{\pi} \int_{\Omega} \frac{g(w)}{s - w} dm(w)$$

solves $\bar{\partial}u = g$ in $\mathbb{C}_{1/2}^+$ and satisfies $\|u\|_{\infty} \leq C_{\Omega} \|g\|_{\infty}$.

We have now collected all our preliminary results and are ready to begin the proof of Theorem 1. For any positive integer N we set $E_N(s) = N^{-s+1/2}$ and consider the space $E_N A_{\beta}$. By a substitution it is evident that any $F \in E_N A_{\beta}$ can be represented

$$F(s) = \int_{\log N}^{\infty} \phi(\xi) e^{-(s-1)/2} d\xi$$

for some $\phi \in L_{\beta}^2[\log N, \infty)$, in view of Lemma 2.

Final step in the proof of Theorem 1. Let us fix $\alpha > 0$ and a bounded sequence $S = (\sigma_j + it_j) \in Z(A_{\beta})$. From this point all constants may depend on α and S . Since S is bounded we may assume $S \subset \Omega(R - 2, \tau - 2)$ for some $R, \tau > 2$. Let Θ be some smooth function defined on $\mathbb{C}_{1/2}^+$ with the following properties:

- Θ is supported on $\Omega(R, \tau)$,
- $\Theta(s) = 1$ for $s \in \Omega(R - 1, \tau - 1)$,
- $|\bar{\partial}\Theta(s)| \leq 2$.

Let $G \in A_{\beta}$ vanish precisely on S and assume furthermore that $\|G\|_{A_{\beta}} = 1$. Now, suppose that $F \in E_N A_{\beta}$, and let $f \in \mathcal{D}_{\alpha}$ be the function obtained by applying Lemma 4 to F , and $\Phi = F - f$. Moreover, let u denote the solution to the equation

$$(7) \quad \bar{\partial}u = \frac{\bar{\partial}(\Theta\Phi)}{GE_N}.$$

The right hand side of (7) is a smooth function compactly supported on $\Omega(R, \tau)$ since $|G(s)|$ is bounded from below where $\bar{\partial}\Theta(s) \neq 0$. We can use Lemma 5 and Lemma 2 to estimate

$$(8) \quad \|u\|_{\infty} \ll \left\| \frac{\bar{\partial}(\Theta\Phi)}{GE_N} \right\|_{\infty} \ll (\log N)^{-1} \|\phi\|_{L_{\beta}^2} = (\log N)^{-1} \|F\|_{A_{\beta}}.$$

We set $T_N F = \Theta\Phi - GE_N u$. The function $T_N F$ has the following properties:

- $T_N F(s) = \Phi(s)$ for $s \in S$,
- $T_N F$ is holomorphic in $\mathbb{C}_{1/2}^+$ since $\bar{\partial}T_N F(s) = 0$ for $s \in \mathbb{C}_{1/2}^+$,
- $T_N F \in E_N A_{\beta}$, by the compact support of Θ and the estimate (8).

Hence T_N defines an operator on $E_N A_{\beta}$. By the triangle inequality, Lemma 4 and the fact that Θ has compact support, it is clear that

$$\|T_N F\|_{A_{\beta}} \leq \|\Theta\Phi\|_{A_{\beta}} + \|GE_N u\|_{A_{\beta}} \ll (\log N)^{-1} \|\phi\|_{L_{\beta}^2} + \|u\|_{\infty} \|G\|_{A_{\beta}}.$$

Since $\|G\|_{A_{\beta}} = 1$ and $\|\phi\|_{L_{\beta}^2} = \|F\|_{A_{\beta}}$ we have $\|T_N\| \ll (\log N)^{-1}$ in view of (8). Let N be large, but arbitrary, and define $F_0(s) = E_N(s)G(s)$. Then $F_0 \in E_N A_{\beta}$ and its norm in this space is ≤ 1 . Set

$$F_j = T_N^j F_0.$$

Let f_j be the Dirichlet series of Lemma 4 obtained from F_j . Then $f_0 + F_1$ vanishes on S , since

$$f_0(s) + F_1(s) = f_0(s) + T_N F_0(s) = f_0(s) + F_0(s) - f_0(s) = F_0(s) = 0,$$

for $s \in S$, by the fact that $T_N F(s) = \Phi(s)$ for $s \in S$. Iteratively, the function $f_0 + f_1 + \cdots + f_j + F_{j+1}$ also vanishes on S . Define

$$f(s) = \sum_{j=0}^{\infty} f_j(s)$$

and choose N so large that $\|T_N\| < 1$ so that $\|F_j\|_{A_\beta} \rightarrow 0$ and, say

$$|f(1)| > \sum_{j=1}^{\infty} |f_j(1)|,$$

so that f is non-trivial in \mathcal{D}_α and vanishing on S . \square

By again following [13], we can modify the iterative scheme in the following way: Let $F \in A_\beta$ be arbitrary, and set $F_0 = F$. Using the algorithm in the same manner as above, we see that $F_1(s) + f_0(s) = F_0(s)$ for $s \in S$. Moreover,

$$F_{j+1}(s) + f_j(s) + f_{j-1}(s) + \cdots + f_0(s) = F(s),$$

for $s \in S$. Continuing as above, we obtain the following result:

Corollary 6. *Suppose $S = (\sigma_j + it_j) \in Z(A_\beta)$ is bounded. For every function $F \in A_\beta$ there is some $f \in \mathcal{D}_\alpha$ such that $f(s) = F(s)$ on S .*

We can extend Theorem 1 and Corollary 6 by considering different weights. Let $w = (w_1, w_2, \dots)$ be a non-negative weight. Define the Hilbert space of Dirichlet series \mathcal{D}_w in the same manner as above, with the added convention that the basis vector n^{-s} is excluded if $w_n = 0$. Recall from [9] that \mathcal{D}_w embeds locally into A_β if and only if

$$(9) \quad \sum_{n \leq x} w_n \ll x(\log x)^\beta,$$

where $\beta > 0$. By modifying the proof of Theorem 1 we can obtain a similar result for \mathcal{D}_w with respect to A_β provided we have

$$(10) \quad \sum_{j^\gamma \leq \log n \leq (j+1)^\gamma} \frac{w_n}{n} \asymp j^{\gamma(\beta-1)-1},$$

as $j \rightarrow \infty$, for some $0 < \gamma < 2/(3 + \beta)$. Several of the weights considered in [9] are possible, but we only mention the case $w_n = (\log n)^\beta$ for $\beta > 0$. These spaces were introduced by McCarthy in [8]. It is easy to show that these weights satisfy (9) and (10) for any $0 < \gamma < 1$, and similar results with respect to A_β are obtained.

Remark. The embeddings of [9] extend to any $\beta \leq 0$, in view of (9), and we get the Hardy space ($\beta = 0$) and Dirichlet spaces ($\beta < 0$) in the half-plane. We can extend the results in [13] in a similar manner as above. However, this is only possible for $-1 \leq \beta < 0$. The method of [13] breaks down for $\beta < -1$ due to the fact that the norms of the corresponding Dirichlet spaces in the half-plane uses higher order derivatives and different estimates are needed.

3. ZEROS OF FUNCTION IN HARDY SPACES OF DIRICHLET SERIES

The Hardy spaces of Dirichlet series \mathcal{H}^p , $1 \leq p \leq \infty$, can be defined as the closure of the set of all Dirichlet polynomials with respect to the norms

$$\left\| \sum_{n=1}^N \frac{a_n}{n^s} \right\|_{\mathcal{H}^p} = \lim_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T \left| \sum_{n=1}^N \frac{a_n}{n^{it}} \right|^p dt \right)^{\frac{1}{p}}.$$

For the basic properties of these spaces, we refer to [4] and [1]. We immediately observe that $\mathcal{H}^2 = \mathcal{D}_0$. In [13], the bounded zero sequences of the spaces \mathcal{H}^p , $2 \leq p \leq \infty$, are studied. In particular, for \mathcal{H}^2 the Blaschke condition

$$(11) \quad \sum_j (\sigma_j - 1/2) < \infty$$

is shown to be both necessary and sufficient. Results for $2 < p < \infty$ are obtained through embeddings $\mathcal{D}_\alpha \subset \mathcal{H}^p \subset \mathcal{H}^2$, where $\alpha < 0$ depends on p . The embedding of \mathcal{H}^p into \mathcal{H}^2 implies that the Blaschke condition (11) is necessary for \mathcal{H}^p .

The sufficient conditions are obtained through a similar result as Theorem 1: For $\alpha < 0$, the spaces \mathcal{D}_α have the same bounded zero sequences as certain weighted Dirichlet spaces in $\mathbb{C}_{1/2}^+$. In particular, for $2 < p < \infty$ there is some $0 < \beta < 1$ such that a sufficient condition for bounded zero sequences of \mathcal{H}^p is

$$(12) \quad \sum_j (\sigma_j - 1/2)^{1-\beta} < \infty,$$

and moreover $\beta \rightarrow 0$ as $p \rightarrow 2^-$. We omit the details, which can be found in [13], where also a complete description for the bounded zero sequences of \mathcal{H}^∞ is contained.

We will consider the case $1 \leq p < 2$, where we have the embeddings

$$(13) \quad \mathcal{H}^2 \subset \mathcal{H}^p \subset \mathcal{D}_\alpha,$$

and $\alpha = 2/p - 1$. That $\mathcal{H}^2 \subset \mathcal{H}^p$ for $1 \leq p < 2$ is trivial, and this shows that (11) is a sufficient condition for bounded zero sequences of \mathcal{H}^p . The other embedding is more interesting: In [5], Helson proved the beautiful inequality

$$(14) \quad \|f\|_{\mathcal{D}_1} = \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{H}^1}.$$

In [9] it is observed that by interpolating between (14) and $\mathcal{H}^2 = \mathcal{D}_0$ one obtains the inequality $\|f\|_{\mathcal{D}_{2/p-1}} \leq \|f\|_{\mathcal{H}^p}$, which provides the required embeddings.

This leads us to consider necessary conditions for zero sequences of the spaces A_β . The zero sequences of Bergman spaces in the unit disc \mathbb{D} have attracted considerable attention. For $\beta > 0$, these are the spaces

$$A_\beta(\mathbb{D}) = \left\{ F \in H(\mathbb{D}) : \|F\| = \int_{\mathbb{D}} |F(z)|^2 (1 - |z|)^{\beta-1} dm(z) < \infty \right\}.$$

Results pertaining to zero sequences of $A_\beta(\mathbb{D})$ are relevant to our case since

$$\phi(s) = \frac{s - 3/2}{s + 1/2}$$

is a conformal mapping from $\mathbb{C}_{1/2}^+$ to \mathbb{D} . Moreover,

$$F \mapsto (s + 1/2)^{-2(\beta+1)} F\left(\frac{s - 3/2}{s + 1/2}\right)$$

defines an isometric isomorphism from $A_\beta(\mathbb{D})$ to A_β . This implies that $S \in Z(A_\beta)$ if and only if $\phi(S) \in Z(A_\beta(\mathbb{D}))$. Theorem 4.1 of [3] shows that the Blaschke condition (11) is both necessary and sufficient provided S is contained in any cone $|t - t_0| \leq c(\sigma - 1/2)$. Unfortunately, the situation becomes more complicated in the general setting and we do not have a Blaschke type condition. The statement analogous to (12) for Bergman spaces is: For every $\epsilon > 0$ and every A_β a necessary condition for zero sequences is

$$\sum_j (\sigma_j - 1/2)^{1+\epsilon} < \infty.$$

Clearly, this condition does not offer any insight into what happens as $p \rightarrow 2^+$. However, using the notion of density introduced by Korenblum in [7] we can provide a generalized condition describing the geometrical information of the zero sequences of $A_\beta(\mathbb{D})$. The most precise results on Korenblum's density are obtained by Seip in [12]. We omit the details, and only mention that this condition in a certain sense tends to (11) when $\beta \rightarrow 0^+$, that is when $p \rightarrow 2^+$.

Remark. The Blaschke condition (11) is well-known to be necessary and sufficient for zero sequences of the Hardy spaces $H^p(\mathbb{C}_{1/2}^+)$. By a theorem in [4], \mathcal{H}^2 embeds locally into $H^2(\mathbb{C}_{1/2}^+)$. This trivially extends to even integers p . Whether the local embedding extends to every $p \geq 1$ is an open question. Observe that if (11) is not the optimal necessary condition for zero sequences of \mathcal{H}^p , $1 \leq p < 2$, then the local embedding would be impossible for these p . Since (12) is a sufficient condition for $p \geq 2$, this does not contradict the local embedding when $p > 2$.

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DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (NTNU), NO-7491 TRONDHEIM, NORWAY

E-mail address: ole.brevig@math.ntnu.no